Natural logarithms of any number (real, imaginary, complex)

$$a + b \cdot i = e^{z}$$

$\ln(a + b \cdot i) = z$ by definition

z can also be a complex number

$$\ln(a + b \cdot i) = c + d \cdot i$$

Starting with original equation

$$a + b \cdot i = e^{c + d \cdot i}$$

Using Euler's relationship, $e^{ix} = \cos(x) + i*\sin(x)$
and law of exponents, $y^{N+M} = y^{N} * Y^{M}$

$$a + b \cdot i = e^{c} \cdot e^{i \cdot d}$$

we can solve for c, observing that e^{c} = radius of the Euler circle on comple› plane

$$e^{c} = \sqrt{a^{2} + b^{2}}$$

$$c = \ln\left(\sqrt{a^{2} + b^{2}}\right)$$

d is the angle that locates the number on the complex plane, or

$d = \operatorname{atan}\left(\frac{b}{a}\right)$ where d can range from 0 to 2π (or $-\pi$ to $+\pi$)

<u>Conclusions</u>

1. Every number (positive, negative, real, imaginary and complex) that can be written as a+bi has a logarithm, except zero. Any number can be written as having a real part, a, and an imaginary part, b x i.

$\text{Real} = R \cdot \cos(\theta)$ $\text{Imaginary} = R \cdot \sin(\theta)$

where

$$\theta = \operatorname{atan}\left(\frac{b}{a}\right)$$

and

$$R = \ln\left(\sqrt{a^2 + b^2}\right)$$

2. Since there are infinitely many roots to atan(b/a), there are infinitely many logartithms for any number. For any number a+b*i, its logarithms are

$$R\cdot\cos(\theta) + i\cdot R\cdot\sin(\theta + N\cdot 2\cdot\pi)$$

Where N is any integer from -∞ to +∞.

Example 1: What are the logarithms of 1?

$R = 0$ [i.e. a^2+b^2 =1; ln(1) = 0]

$$\theta = 0 + M\cdot 2\cdot\pi$$

$$\ln(1) = 0 + 2\cdot\pi\cdot i$$

also

$$\ln(1) = 4\cdot\pi\cdot i$$

etc...

Example 2. What is the logarithm of i (sqrt(-1))?

Again, R=0, but θ=π/2 (or 90 degrees)

$$\ln(i) = 0 + i\cdot\frac{\pi}{2}$$

also

$$\ln(i) = i\cdot\frac{5\cdot\pi}{2}$$

etc ...

Example 3. So now you are wondering, what is the logarithm of -1? We didn't learn that in calculus

Again R=1, but now θ=π, or 180 degrees (sort of a weird thing, but we are taking the atan of (-1))

$$\ln(-1) = i\cdot\pi$$

also

$$\ln(i) = i\cdot 3\cdot\pi$$

let's try it out

$$e^{i\cdot 3\cdot\pi} = -1$$

$$\ln(-1) = 3.142i$$

$$i := \sqrt{-1}$$